2010 American Control Conference
Marriott Waterfront, Baltimore, MD, USA
June 30-July 02, 2010

Natural Entrainment of Collocated Mechanical Systems via Decentralized Multi-agent Feedback

Y. Futakata
Information Physics & Computing
The University of Tokyo
futakata_yoshiaki@ipc.i.u-tokyo.ac.jp

T. Iwasaki
Mechanical & Aerospace Engineering
University of California, Los Angeles
tiwasaki@ucla.edu

Abstract— **This paper considers the design of feedback controllers for a class of collocated mechanical systems, aiming to achieve a natural mode of oscillation as a stable limit cycle of the closed-loop system. Motivated by recent results on central pattern generators, the controller is formed as a collection of multiple identical agents without direct communications to each other. Each agent is described as a transfer function followed by a static nonlinearity, and is placed between a sensor/actuator pair. Several simple linear dynamics are considered for the agents. The method of multivariable harmonic balance (MHB) is used to obtain approximate conditions for achieving entrainment to a natural oscillation, and numerical examples suggest that the proposed design conditions are fairly reliable in spite of sinusoidal approximations adopted in the MHB method. It is found that a band-pass filtering property is essential for the linear dynamics of each agent to achieve entrainment to an arbitrarily specified mode of natural oscillation.**

I. Introduction

Resonance, or natural motion in general, can often be exploited for increased efficiency when controlling a mechanical system to achieve oscillatory behaviors. Rhythmic body movements during animal locomotion are known to be controlled by central pattern generators (CPGs), a group of neurons that are interconnected in a specific manner [1], [2]. Recent studies on biological control systems have shown that feedback controllers with architectures borrowed from CPGs can achieve entrainment to the resonance of a single degree-of-freedom (DOF) mechanical system [3]–[7], and more generally, entrainment to a natural mode of oscillation for multi-DOF systems [8], [9]. While the dynamics of a CPG, which is a nonlinear oscillator by itself, can be fairly complex, it is suggested [3], [7] that the mechanisms underlying the natural entrainment are rather simple — locally negative damping supplied through positive rate feedback or negative integral feedback, with saturation.

To explain the idea, consider a single-DOF system

$$m\ddot{x} + d\dot{x} + kx = u$$

where m, k, and d are the mass, stiffness, and damping, $x(t)$ is the displacement, and $u(t)$ is the force input. The objective is to design a feedback controller to achieve an oscillation at the natural frequency $\sqrt{k/m}$. The simplest solution would be $u(t) = d\dot{x}$ to cancel the damping, but this does not result in structurally stable oscillation. The idea of positive rate feedback with saturation is to use the control law $u = \psi(\dot{x})$ where ψ is a saturation nonlinearity with the slope at the origin greater than d. Such feedback will provide locally negative damping to destabilize the origin. On the other hand, the saturation effect will reduce the negative damping in the region away from the origin, making every trajectories bounded. As a result, the trajectory converges to a limit cycle on which the negative damping supplied by the control input just cancels the mechanical damping, leading to an oscillation at (or near) the natural frequency. The idea of negative integral feedback with saturation is less obvious in the time domain, but it is easy to see that $u = -\psi(\int x dt)$ would have an effect similar to $u = \psi(\dot{x})$ in the frequency domain, at least for the case where ψ is a linear function.

Motivated by the natural entrainment results, this paper explores the potential of multi-agent controllers with simple nonlinear dynamics to achieve entrainment to a prescribed mode of natural oscillation. In particular, we consider the class of mechanical systems arising from body biomechanics, where stiffness, damping, sensors, and actuators are all collocated. An "agent" is described by $u_i = \mathrm{g}_i\psi(r(s)h_i y_i)$ and is placed between each mechanical (input,output) pair (u_i, y_i) as a feedback controller, where g_i and h_i are the actuator and sensor gains, $r(s)$ is a single-input-single-output (SISO) transfer function, and ψ is a saturation-type nonlinearity. We consider the differentiator and integrator for $r(s)$, as well as their extensions to the low-pass, high-pass, and band-pass filters. We shall investigate whether a set of agents, without direct communications to each other, is able to achieve natural entrainment, in line with a previous research [8] that revealed natural entrainment capability of decentralized CPGs. A systematic method will be developed for designing decentralized multi-agent controllers so that the multivariable harmonic balance (MHB) equation [10], [11] admits a solution at or near one of the natural modes of oscillation. Numerical examples will then validate the accuracy and reliability of the proposed design method. All proofs except the one for Theorem 1 are omitted due to the page limitation.

We use the following notation. Let $\jmath := \sqrt{-1}$. The symbols $\mathbb{R}_+$, and $\mathbb{I}_n$ denote the set of real positive numbers and the set of integers from 1 to n, respectively. For a vector $v \in \mathbb{R}^n$ with the i^{th} entry v_i, the $n \times n$ diagonal matrix whose (i,i) entry is v_i, is denoted by $\mathrm{diag}(v)$ or $\mathrm{diag}(v_1, \cdots, v_n)$.

This work is supported in part by NSF No.0654070, by ONR MURI Grant N00014-08-1-0642, and by The Ministry of Education, Science, Sport, and Culture, Japan under Grant 21246067.

For sets X and Y, the set X/Y is defined as a set of elements which belong to X but not to Y.

II. Problem Formulation and Approach

A. Problem Statement

Consider the class of mechanical systems given by

$$J\ddot{x} + D\dot{x} + Kx = Bu, \quad y = Cx, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, and $u(t), y(t) \in \mathbb{R}^m$ are the generalized coordinates, inputs, and outputs, respectively, and J, D, and K are the inertia, damping, and stiffness matrices, respectively. Recall that a natural mode of the system (1) is defined by a pair of the natural frequency $\omega_\ell \in \mathbb{R}_+$ and mode shape $\xi_\ell \in \mathbb{R}^n$ satisfying $(K - \omega_\ell^2 J)\xi_\ell = 0$ with $\xi_\ell \neq 0$. There are n natural modes, (ω_ℓ, ξ_ℓ) with $\ell \in \mathbb{I}_n$, for the n-DOF system. We define the ℓ^{th} mode shape of the output $\hat{y}_\ell := C\xi_\ell$ and denote its i^{th} entry by $\hat{y}_{\ell i}$. The following assumption is enforced throughout the paper.

Assumption 1: The matrices B and C are square ($m = n$) nonsingular, and satisfy $C = B^{\mathsf{T}}$. The matrices J, D, and K are symmetric positive definite. The stiffness matrix is given by

$$K = B\mathcal{K}C, \quad \mathcal{K} := \mathrm{diag}(k_1, \cdots, k_n).$$

for some scalars $k_i \in \mathbb{R}_+$ with $i \in \mathbb{I}_n$. The damping matrix is given by $D = \rho K$ for some $\rho \in \mathbb{R}_+$. The natural frequencies are distinct and are arranged in the ascending order: $0 < \omega_1 < \omega_2 < \cdots < \omega_n$. During each natural mode of oscillation, no output variable remains stationary: $\hat{y}_{\ell i} \neq 0$ for all $\ell, i \in \mathbb{I}_n$.

The conditions in Assumption 1 are motivated by biological systems where the actuators, sensors, and stiffness and damping elements are all located at the same position. In addition, Rayleigh damping $D = \rho K$ and distinct natural frequencies are assumed so that the analysis becomes simple enough to provide insights into the natural entrainment mechanism. We shall refer to the system (1) satisfying Assumption 1 as the *collocated* system.

We consider a set of n identical agents to control the system. Each agent is an SISO system described as a linear transfer function $r(s)$ followed by a static nonlinearity ψ, and is placed between each sensor/actuator pair as a feedback controller:

$$u_i = \mathrm{g}_i \psi(q_i), \quad q_i = r(s) h_i y_i$$

where g_i and h_i are feedback gains, and q_i is an internal variable. The static nonlinearity ψ is a sigmoid function having the following properties:

- ψ is odd, bounded, and strictly increasing.
- $\psi(x)$ is strictly concave on $x > 0$, and $\psi'(0) = 1$.

We denote by Ψ the class of functions satisfying these conditions. For example, $\tanh(x)$ belongs to this class. The set of n agents can be written as

$$u = G\Psi(q), \quad q = r(s)Hy, \tag{2}$$

where $\mathrm{g}, h \in \mathbb{R}^n$ and $q(t) \in \mathbb{R}^n$ are the vectors defined by stacking g_i, h_i, and $q_i(t)$ in columns, respectively, and

$$G := \mathrm{diag}(\mathrm{g}), \quad H := \mathrm{diag}(h), \quad \Psi := \psi I.$$

To seek an essential mechanism for natural entrainment, we consider five different types of the transfer function $r(s)$:

D	I	LPF	HPF	BPF
s	$\frac{1}{s}$	$\frac{\omega_o}{s+\omega_o}$	$\frac{s}{s+\omega_o}$	$\frac{2\omega_o s}{(s+\omega_o)^2}$

where D, I, LPF, HPF and BPF stand for the differentiator, integrator, low-pass filter, high-pass filter and band-pass filter, respectively. Throughout the paper, we restrict our attention to the case where ω_o is positive, g_i and h_i are nonzero for all $i \in \mathbb{I}_n$, and $\psi \in \Psi$. In addition, we consider positive h_i only, as its sign can be absorbed into g_i. Let us denote by $\mathscr{R}$ the collection of the five transfer functions.

With the controller (2) and the mechanical system (1), the closed-loop system is given by $\mathcal{G}(s, \Psi)z = 0$ where

$$\mathcal{G}(s, \Psi) := \begin{bmatrix} Js^2 + Ds + K & -BG\Psi \\ r(s)HC & -I \end{bmatrix}, \ z := \begin{bmatrix} x \\ q \end{bmatrix}. \tag{3}$$

We would like to solve the following problems posed on (3):

1) Characterize design conditions on the controller such that the closed-loop system has a prescribed mode of natural oscillation as a stable limit cycle.
2) Compare the design conditions for the five transfer functions $r(s) \in \mathscr{R}$, and determine what agent dynamics is essential for achieving natural entrainment.

These problems will be addressed in an approximate setting as described in the next section.

B. Multivariable Harmonic Balance

Assume that the closed-loop system (3) has a periodic solution $z(t)$ with a frequency ω. We shall approximate $z(t)$ by a sinusoidal signal $z(t) = \Re[\hat{z}e^{\jmath\omega t}]$, where $\hat{z} \in \mathbb{C}^{2n}$ is the phasor of $z(t)$. The nonlinearity ψ may be approximated by the describing function when its input signal is a sinusoid:

$$\psi(v) \cong \kappa(\alpha)v \quad \text{for} \quad v := \alpha \sin\omega t,$$
$$\kappa(\alpha) := \frac{4}{\alpha\pi}\int_0^{\pi/2} \psi(\alpha\sin\theta)\sin\theta d\theta. \tag{4}$$

It was shown in [7] that $\kappa(\alpha)$ for $\psi \in \Psi$ takes a value in the interval $(0, 1)$ when $\alpha > 0$, and κ is strictly decreasing from 1 to 0 on $\alpha > 0$.

With the sinusoidal approximations and describing functions, the equation for the closed-loop system (3) reduces to the multivariable harmonic balance (MHB) equation:

$$\mathcal{G}(j\omega, \Phi)\hat{z} = 0, \tag{5}$$

where Φ is an amplitude-dependent matrix gain that approximates the nonlinear function Ψ for sinusoidal inputs:

$$\Phi := \mathrm{diag}(\kappa_1, \cdots, \kappa_n), \quad \kappa_i := \kappa(|\hat{q}_i|),$$

where $\hat{q}_i$ is the phasor of $q_i(t)$. For a set of fixed amplitudes $|\hat{q}_i|$, let us define the associated linear system by $\mathcal{G}(s,\Phi)z = 0$, or equivalently,

$$\begin{aligned} &A(s)x = 0, \\ &A(s) := Js^2 + Ds + K - r(s)BG\Phi HC. \end{aligned} \tag{6}$$

The basic idea of the MHB method [10] is the following.

- If $(\omega, \hat{z})$ satisfies (5), then the system (3) is expected to possess oscillatory trajectory $z(t) \cong \Re[\hat{z}e^{\jmath\omega t}]$.
- The estimated oscillation is expected to be stable if the associated linear system (6) is marginally stable with $\pm\jmath\omega$ being the only eigenvalues on the imaginary axis.

We call $(\omega, \hat{z})$ a *stable solution* of (5) if it satisfies (5), and (6) satisfies the marginal stability condition. The controller (2) is said to *assign* $(\omega, \hat{x})$ a (stable) oscillation profile if $(\omega, \hat{x}, \hat{q})$ is a (stable) solution of (5) for some $\hat{q}$.

III. Natural Entrainment Analysis

A. The MHB condition

Consider the mechanical system (1) driven by the feedback controller (2). We would like to characterize the set of controller parameters for which the MHB equation (5) has a stable solution $(\omega, \hat{x}) = (\omega_\ell, \xi_\ell)$ at a prescribed mode $\ell \in \mathbb{I}_n$. However, it turns out that such set is empty except for a special case (I with $\ell = 1$) and therefore we consider a relaxation of the natural entrainment condition. In particular, we enforce the mode shape but not the frequency, and seek a stable solution $(\omega, \hat{x})$ such that $\hat{x} = \xi_\ell$ and $\omega \cong \omega_\ell$.

First note that the phasor equation for the mechanical system is given by

$$(K + \jmath\omega D - \omega^2 J)\hat{x} = B\hat{u}, \quad \hat{y} = C\hat{x}.$$

When $\hat{x} = \xi_\ell$, this reduces to

$$\left[\left(1 - \left(\frac{\omega}{\omega_\ell}\right)^2\right)K + \jmath\omega D\right]\xi_\ell = B\hat{u}, \quad \hat{y} = C\xi_\ell.$$

Exploiting the collocated structure,

$$\left[1 - \left(\frac{\omega}{\omega_\ell}\right)^2 + \jmath\omega\rho\right]k_i\hat{y}_{\ell i} = \hat{u}_i.$$

Thus, the mechanical system imposes the following input-output relationship when the mechanical variables oscillate sinusoidally with a mode shape ξ_ℓ:

$$\frac{\hat{y}_{\ell i}}{\hat{u}_i} = \frac{p_\ell(\omega)}{k_i}, \quad p_\ell(\omega) := \frac{\omega_\ell}{2\omega(\jmath\zeta_\ell - \varpi_\ell)}, \tag{7}$$

$$\zeta_\ell := \frac{\rho\omega_\ell}{2}, \quad \varpi_\ell := \frac{1}{2}\left(\frac{\omega}{\omega_\ell} - \frac{\omega_\ell}{\omega}\right). \tag{8}$$

For later reference, we define ζ_ℓ and ϖ_ℓ for $\ell \in \mathbb{I}_n$ and $\ell = o$. The phasor equation for the controller, after quasi-linearization via describing functions, is given by

$$\hat{u} = G\Phi\hat{q}, \quad r(\jmath\omega)HC\hat{x} = \hat{q},$$

or equivalently,

$$\frac{\hat{u}_i}{\hat{y}_i} = \mathrm{g}_i h_i \kappa(|\hat{q}_i|) r(\jmath\omega), \quad \hat{q}_i = r(\jmath\omega)h_i\hat{y}_i. \tag{9}$$

From (7) and (9), the MHB equation is given by

$$\mathrm{g}_i h_i \kappa(|\hat{q}_i|) p_\ell(\omega) r(\jmath\omega)/k_i = 1.$$

We denote the real part and imaginary parts of $1/r(\jmath\omega)$ and $1/(p_\ell(\omega)r(\jmath\omega))$ by

$$\sigma + \jmath\lambda := \frac{1}{r(\jmath\omega)}, \quad \phi_\ell + \jmath\theta_\ell := \frac{1}{p_\ell(\omega)r(\jmath\omega)}. \tag{10}$$

For each case of $r(s)$, the parameters σ and λ are given by

	D	I	LPF	HPF	BPF
σ	0	0	1	1	1
λ	$-\dfrac{1}{\omega}$	ω	$\dfrac{\omega}{\omega_o}$	$-\dfrac{\omega_o}{\omega}$	ϖ_o

(11)

The following result summarizes the above development and provides a necessary and sufficient condition for satisfaction of the MHB equation in a form suitable for analysis and synthesis.

Lemma 1: Consider the mechanical system (1) and controller (2), where $r(s)$ is an arbitrary transfer function that possibly depends on a parameter ω_o. Let $\ell \in \mathbb{I}_n$, $\omega \in \mathbb{R}_+$, and $\hat{q} \in \mathbb{C}^n$ be given. Assume σ and λ in (10) are well defined. The MHB equation (5) with $\hat{x} = \xi_\ell$ is satisfied if and only if

$$\eta_i \kappa(|\hat{q}_i|) = \phi_\ell(\omega, \omega_o), \tag{12}$$

$$\theta_\ell(\omega, \omega_o) = 0, \tag{13}$$

$$\hat{q}_i = r(\jmath\omega)h_i\hat{y}_{\ell i}, \tag{14}$$

hold for all $i \in \mathbb{I}_n$, where κ is defined by (4), $\hat{y}_{\ell i}$ is the i^{th} entry of $\hat{y}_\ell := C\xi_\ell$, and

$$\begin{aligned} \eta_i &:= \mathrm{g}_i h_i/k_i, \\ \phi_\ell(\omega, \omega_o) &:= -(2\omega/\omega_\ell)(\sigma\varpi_\ell + \lambda\zeta_\ell), \\ \theta_\ell(\omega, \omega_o) &:= (2\omega/\omega_\ell)(\sigma\zeta_\ell - \lambda\varpi_\ell). \end{aligned}$$

For a given controller, the frequency of oscillation with a mode shape ξ_ℓ, if any, can be estimated as a solution to (13). The estimated frequency, denoted by w_ℓ, for specific choices of $r(s)$ is summarized in Table I. We see that w_ℓ is exactly equal to ω_ℓ when $r(s)$ is D or I. For LPF (respectively HPF), w_ℓ is greater (smaller) than ω_ℓ, but is close to ω_ℓ when the damping ρ is small and/or ω_o is small (large). For BPF, condition (13) reduces to $\zeta_\ell = \varpi_o\varpi_\ell$, the right hand side of which can be viewed as a function of ω that vanishes at $\omega = \omega_o$ and ω_ℓ and is convex on $\omega > 0$. Hence, (13) has two distinct real positive solutions, denoted by $\omega = \mathrm{w}_\ell^{\pm}$, such that ω_o and ω_ℓ are contained in the interval $\mathrm{w}_\ell^{-} < \omega < \mathrm{w}_\ell^{+}$. When the damping ζ_ℓ is small, one of $\mathrm{w}_\ell^{\pm}$ is close to ω_o, and the other is close to ω_ℓ. Explicit formulas for $\mathrm{w}_\ell^{\pm}$ are easily derived, but are not shown for brevity.

Condition (12) suggests a bound on the feedback gain η_i so that the MHB equation admits a solution and predicts an oscillation at (ω, ξ_ℓ). In particular, the describing function

TABLE I

ESTIMATED FREQUENCY AND GAIN BOUND

$r(s)$	w_ℓ	$\phi_\ell(\mathrm{w}_\ell, \omega_o)$
D	ω_ℓ	ρ
I	ω_ℓ	$-\rho\omega_\ell^2$
LPF	$\omega_\ell\sqrt{1+\rho\omega_o}$	$-\rho\omega_o(1+(\mathrm{w}_\ell/\omega_o)^2)$
HPF	$\omega_\ell/\sqrt{1+\rho\omega_\ell^2/\omega_o}$	$\rho\omega_o(1+(\mathrm{w}_\ell/\omega_o)^2)$

$\kappa(|\hat{q}_i|)$ takes a value in the interval $0 < \kappa(|\hat{q}_i|) < 1$, and hence (12) implies $0 < \phi_\ell(\omega, \omega_o)/\eta_i < 1$. Thus, if the controller assigns (ω, ξ_ℓ), then it is necessary (and sufficient) that η_i have the same sign as $\phi_\ell(\omega, \omega_o)$ and its absolute value be greater than $|\phi_\ell(\omega, \omega_o)|$.

Theorem 1: Consider the mechanical system (1) and controller (2), where $\psi \in \Psi$ and $r(s) \in \mathscr{R}$. Let $\ell \in \mathbb{I}_n$ and $\omega \in \mathbb{R}_+$ be given. The controller assigns (ω, ξ_ℓ) if and only if there exists $\eta \in \mathbb{R}^n$ such that

$$\mathrm{g}_i = \frac{k_i \eta_i}{h_i}, \quad h_i = \frac{\alpha_i}{|r(j\omega)\hat{y}_{\ell i}|}, \tag{15}$$

where α_i is the real positive number satisfying $\kappa(\alpha_i) = \phi_\ell(\omega, \omega_o)/\eta_i$, and, for all $i \in \mathbb{I}_n$,

(i) $\omega = \mathrm{w}_\ell$ and $\eta_i > \phi_\ell(\omega, \omega_o)$ for D and HPF;

(ii) $\omega = \mathrm{w}_\ell$ and $\eta_i < \phi_\ell(\omega, \omega_o)$ for I and LPF;

(iii) $\omega = \mathrm{w}_\ell{}^-$ and $\eta_i > \phi_\ell(\omega, \omega_o)$, or $\omega = \mathrm{w}_\ell{}^+$ and $\eta_i < \phi_\ell(\omega, \omega_o)$, for BPF.

For each case, the sign of the gain bound $\phi_\ell(\omega, \omega_o)$ is such that $\eta_i = 0$ is excluded from the feasible set of η_i.

Proof: From Lemma 1, the controller assigns (ω, ξ_ℓ) if and only if (12) and (13) are satisfied with $\hat{q}_i$ defined by (14). Following the preceding development, one can verify that (12) and (13) are satisfied if and only if the given conditions are satisfied. ∎

Theorem 1 suggests that (η, ω_o) are the essential controller parameters where $\eta := [\eta_1, \cdots, \eta_n]^\mathsf{T}$. The frequency ω of the closed-loop oscillation is determined by ω_o through (13). For a given η, the parameters (g, h) that assign (ω, ξ_ℓ) are obtained by solving (12) for $|\hat{q}_i|$ $(= \alpha_i)$, and (14) for h_i, and finally letting $\mathrm{g}_i := k_i\eta_i/h_i$.

B. Stability Analysis and Existence of Oscillations

The quasi-linear system (6) depends on the product of g_i, h_i and $\kappa(\alpha_i)$, but not on the individual values, and α_i is determined by (η, ω_o) for a chosen mode ℓ through Theorem 1. Hence, the stability property of (6) is determined by ℓ and (η, ω_o). Due to the special structure of (1) satisfying Assumption 1, pole locations of the associated linear system (6) can be examined through the Routh stability criterion. The following result shows when (6) satisfies the marginal stability condition.

Theorem 2: Consider the mechanical system (1) and controller (2), where $\psi \in \Psi$ and $r(s) \in \mathscr{R}$. Let $\ell \in \mathbb{I}_n$ and $\omega \in \mathbb{R}_+$ be given and suppose the controller assigns (ω, ξ_ℓ). Then the following stability property holds for each controller:

(i) All the poles of (6) are on the imaginary axis for D;

(ii) All the poles except $s = \pm j\omega$ of (6) are in the open left half plane if and only if $\ell = 1$, for I, LPF, and HPF;

(iii) All the poles except $s = \pm j\omega$ of (6) are in the open left half plane if and only if

$$\phi_i(\mathrm{w}_i^+, \omega_o) < \phi_\ell(\omega, \omega_o) < \phi_i(\mathrm{w}_i^-, \omega_o) \tag{16}$$

for all $i \in \mathbb{I}_n/\{\ell\}$ where $\mathrm{w} = \mathrm{w}_i^\pm$ are the positive solutions of $\theta_i(\mathrm{w}, \omega_o) = 0$.

In the case of D, all the poles of (6) are on the imaginary axis, and our method does not suggest whether the assigned oscillation (ω, ξ_ℓ) is stable. For I, LPF and HPF, Theorem 2 shows that only the first natural mode can be stably assigned. For BPF, the condition in (16) means that the selected mode $\ell \in \mathbb{I}_n$ can be stably assigned if and only if it gives the the smallest bound $|\phi_\ell(\omega, \omega_o)|$ on the magnitude of the feedback gain $|\eta|$, where we note that $\phi_i(\mathrm{w}_i^+, \omega_o) < 0$ and $\phi_i(\mathrm{w}_i^-, \omega_o) > 0$ always hold.

Theorem 1 characterizes the set of controllers that assign a given oscillation profile (ω, ξ_ℓ) as a solution to the MHB equation. Theorem 2 provides additional conditions under which the assignment is done in a stable manner. In this way, one can design a controller so that the closed-loop system is likely to have an oscillation at (ω, ξ_ℓ). However, existence of such oscillation is not guaranteed due to the approximations involved with the MHB method. The following result shows that the existence of Y-oscillation [10], [12], [13] is indeed guaranteed when the same value is used for the feedback gains η_i in every input-output channel, under a mild assumption on the nature of the equilibrium. Note that Y-oscillation is less restrictive than a periodic trajectory; that is, a periodic trajectory is a Y-oscillation but the converse is not necessarily true.

Theorem 3: Let a mechanical system (1) and controller (2) be given, where $\psi \in \Psi$ and $r(s) \in \mathscr{R}$. Suppose the origin of the closed-loop system (3) is a hyperbolic equilibrium, and there exist $\eta_o \in \mathbb{R}$ such that $\mathrm{g}_i h_i = k_i \eta_o$ holds for all $i \in \mathbb{I}_n$, and $\eta_i := \eta_o$ satisfies statements (i)–(iii) in Theorem 1 for some $\ell \in \mathbb{I}_n$. Then, for almost all initial conditions, the trajectory $z(t)$ of (3) oscillates in the steady state.

This theorem is proved by showing that (a) the equilibrium at the origin is unique and unstable, and (b) every trajectory is bounded. The requirement that $\eta_i := \mathrm{g}_i h_i/k_i$ be independent of i is introduced for the sake of tractability, and hence may not be necessary to conclude existence of oscillation. However, such choice may be reasonable because it achieves signal conditioning by making the magnitude of the input signal q_i to the nonlinearity ψ uniform over every agents.

In summary, a controller can be designed to assign (ω, ξ_ℓ) as a (stable) oscillation profile by first selecting a natural mode ℓ and parameters (ω_o, η_o) so that $0 < \phi_\ell(\omega, \omega_o)/\eta_o < 1$ where $\omega := \mathrm{w}_\ell{}^+$ or $\mathrm{w}_\ell{}^-$ for BPF, and $\omega := \mathrm{w}_\ell$ for the others, and then setting the feedback gains g_i and h_i by (15). This general procedure applies to all the cases of $r(s) \in \mathscr{R}$, with the following specific details:

- For D, the natural mode (ω_ℓ, ξ_ℓ) is assigned exactly when $\eta_o > \rho$, but the resulting oscillation may not be stable.

- For I, the natural mode (ω_ℓ, ξ_ℓ) is assigned exactly when $\eta_o < -\rho\omega_\ell^2$, and the resulting oscillation is expected to be stable when the first mode ($\ell = 1$) is selected.
- For LPF and HPF, the natural mode (ω_ℓ, ξ_ℓ) is assigned approximately ($\omega \cong \omega_\ell$) when

$$\begin{array}{ll} \text{LPF:} & \eta_o > -\rho\omega_o(1 + (\mathrm{w}_\ell/\omega_o)^2), \\ \text{HPF:} & \eta_o < \quad \rho\omega_o(1 + (\mathrm{w}_\ell/\omega_o)^2), \end{array} \tag{17}$$

and the resulting oscillation is expected to be stable when the first mode ($\ell = 1$) is selected.
- For BPF, the natural mode (ω_ℓ, ξ_ℓ) is assigned approximately ($\omega \cong \omega_\ell$) when either

$$\begin{array}{lll} \eta_o < \phi_\ell(\mathrm{w}_\ell{}^+, \omega_o), & \omega_o \le \omega_\ell, & \text{or} \\ \eta_o > \phi_\ell(\mathrm{w}_\ell{}^-, \omega_o), & \omega_o \ge \omega_\ell. & \end{array} \tag{18}$$

The oscillation profile (ω_o, ξ_ℓ) is assigned approximately ($\omega \cong \omega_o$) when either

$$\begin{array}{lll} \eta_o < \phi_\ell(\mathrm{w}_\ell{}^+, \omega_o), & \omega_o \ge \omega_\ell, & \text{or} \\ \eta_o > \phi_\ell(\mathrm{w}_\ell{}^-, \omega_o), & \omega_o \le \omega_\ell. & \end{array} \tag{19}$$

In each case, the resulting oscillation is expected to be stable when (16) holds with $\omega = \mathrm{w}_\ell{}^+$ for negative η_o or $\mathrm{w}_\ell{}^-$ for positive η_o.

In the above, the approximate assignment ($\omega \cong \omega_\ell$ or $\omega \cong \omega_o$) has smaller error if the damping ρ is smaller.

The control designs for D and I are simple; just choose a positive (for D) or negative (for I) feedback gain η_o with its magnitude large enough. For LPF, HPF, and BPF, the design parameters (ω_o, η_o) can be selected using the mode-partition diagram as shown in Fig. 1. In this diagram, the parameter plane (ω_o, η_o) can be partitioned into multiple regions $\mathbb{O}_{\ell\ell}$, $\mathbb{O}_{o\ell}$, and $\mathbb{O}_{oo}$, where oscillation profiles (ω_ℓ, ξ_ℓ) and (ω_o, ξ_ℓ) are assigned approximately but stably in $\mathbb{O}_{\ell\ell}$ and $\mathbb{O}_{o\ell}$, respectively, and no oscillation is assigned stably in $\mathbb{O}_{oo}$ because the origin is stable. Specifically, for LPF and HPF, $\mathbb{O}_{11}$ is defined by (17) with $\ell = 1$ and the complement of this region is $\mathbb{O}_{oo}$. For BPF, $\mathbb{O}_{\ell\ell}$ is defined by (18) and (16), $\mathbb{O}_{o\ell}$ is defined by (19) and (16), and the remaining region is $\mathbb{O}_{oo}$.

IV. Design Examples

We give design examples to illustrate the proposed method for natural entrainment and to complement the approximate nature of the theoretical conditions using numerical simulations. We consider a 3-link mechanical arm with flexible joints on a horizontal plane, where the first link is connected to the inertial frame through a rotational joint. The i^{th} link has uniformly distributed mass m_i, length ℓ_i, and the moment of inertia $J_i := m_i\ell_i^2/12$. Mounted at the i^{th} joint are a spring of stiffness k_i, a dashpot of damping coefficient ρk_i, and an actuator that generates torque input u_i. The equation of motion linearized at the origin is given by (1) where x_i is the angular displacement of the i^{th} link, y_i is the relative angle

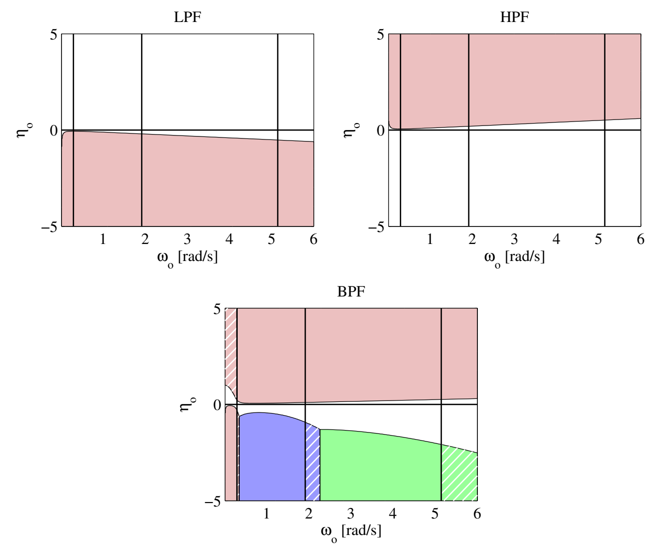

Fig. 1. Mode-partition Diagrams for 3-link Mechanical Arm: ($\omega_1 = 0.292$, $\omega_2 = 1.92$, $\omega_3 = 5.15$); solid pink: $\mathbb{O}_{11}$, shaded pink: $\mathbb{O}_{o1}$, solid blue: $\mathbb{O}_{22}$, shaded blue: $\mathbb{O}_{o2}$, solid green: $\mathbb{O}_{33}$, shaded green: $\mathbb{O}_{o3}$, where in the region labeled as $\mathbb{O}_{pq}$, the MHB analysis predicts that the closed-loop oscillation has a frequency close to the p^{th} natural frequency ω_p (or ω_o if $p = 0$) with the q^{th} natural mode shape ξ_q.

between the i^{th} and $(i+1)^{\text{th}}$ link, and

$$\begin{aligned} &J := \mathrm{diag}(J_1, J_2, J_3) + L^\mathsf{T} M L, \\ &M := \mathrm{diag}(m_1, m_2, m_3), \quad \mathcal{K} := \mathrm{diag}(k_1, k_2, k_3), \\ &C := B^\mathsf{T}, \quad K := B\mathcal{K}C, \quad D := \rho K, \\ &B := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \quad L := \begin{bmatrix} \ell_1/2 & 0 & 0 \\ \ell_1 & \ell_2/2 & 0 \\ \ell_1 & \ell_2 & \ell_3/2 \end{bmatrix}. \end{aligned}$$

We use the following parameters for the mechanical system:

$$\ell_i = 1, \quad m_i = 1, \quad k_i = 1, \quad \rho = 0.1,$$

for $i \in \mathbb{I}_3$. The natural modes are given by

ℓ	1^{st} mode	2^{nd} mode	3^{rd} mode
ω_ℓ	0.292	1.92	5.15
ξ_ℓ	$\begin{bmatrix} 0.402 \\ 0.615 \\ 0.678 \end{bmatrix}$	$\begin{bmatrix} 0.476 \\ -0.242 \\ -0.846 \end{bmatrix}$	$\begin{bmatrix} 0.291 \\ -0.654 \\ 0.698 \end{bmatrix}$

where the mode shape ξ_ℓ is normalized so that $\|\xi_\ell\| = 1$.

For the control design, we use the nonlinearity $\psi(x) = \tanh(x)$. The feedback gain η_o is selected to satisfy the bound $\eta_o > \rho$ and $\eta_o < -\rho\omega_\ell^2$ for D and I, respectively, and (ω_o, η_o) is selected from the mode-partition diagrams in Fig. 1 for LPF, HPF, and BPF. The feedback gains (g, h) are then calculated from (15). For each controller thus designed, numerical simulations are performed for the system and accuracy of the MHB method is evaluated with the following error criteria:

$$\begin{aligned} &e_{\text{PRD}} := |\omega_{\text{MHB}} - \omega_{\text{SIM}}|/\omega_{\text{SIM}}, \quad e_\omega := |\omega_{\text{SIM}} - \omega_{\text{NAT}}|/\omega_{\text{NAT}}, \\ &e_x := \|\bar{x}_{\text{SIM}} - x_{\text{NAT}}\|/\|x_{\text{NAT}}\|, \quad \bar{x}_{\text{SIM}} := x_{\text{SIM}}/\|x_{\text{SIM}}\|, \end{aligned}$$

where ω_{MHB}, ω_{SIM}, and ω_{NAT} are the oscillation frequency predicted by the MHB method, the actual frequency achieved in the simulations, and the natural frequency targeted in the design, respectively, and x_{SIM} and x_{NAT} are defined similarly for the mode shape. Below, the results for the error values are reported in the unit of %.

For D, we designed three controllers, one for each mode, and the result is summarized below.

	ξ_ℓ	η	ω_{MHB}	ω_{SIM}	e_{PRD}	e_ω	e_x
(a)	ξ_1	3	0.292	1.90	84	551	175
(b)	ξ_2	3	1.92	1.91	0.4	0.4	0.0
(c)	ξ_3	3	5.15	4.99	3.2	3.1	0.0

While our theoretical result is inconclusive for D regarding which mode can be stably entrained, numerical simulations revealed that the second and third modes could be entrained but not the first mode.

For I, we obtained the following result.

ξ_ℓ	η	ω_{MHB}	ω_{SIM}	e_{PRD}	e_ω	e_x
ξ_1	-3	0.292	0.292	0.0	0.0	0.0

As predicted in Theorem 2, only the first mode can be stably entrained. The errors e_{PRD}, e_ω and e_x are all small, indicating accuracy of the MHB method.

For LPF, we designed five controllers with different values of ω_o and obtained the following results.

	ξ_ℓ	ω_o	η	ω_{MHB}	ω_{SIM}	e_{PRD}	e_ω	e_x
(a)	ξ_1	0.2	-3	0.295	0.295	0.0	1.0	0.0
(b)	ξ_1	1.0	-3	0.307	0.307	0.0	4.8	0.0
(c)	ξ_1	4.0	-3	0.346	0.344	0.6	17	0.0
(d)	ξ_1	6.0	-3	0.370	0.367	0.9	25	0.0

In accordance with the mode-partition diagram in Fig. 1, the first mode was entrained regardless of the value of ω_o. The errors e_{PRD} and e_x were found to be small, but ω_{SIM} deviates from ω_1 as ω_o gets larger as predicted in Table I.

For HPF, we designed four controllers with the results given below.

	ξ_ℓ	ω_o	η	ω_{MHB}	ω_{SIM}	e_{PRD}	e_ω	e_x
(a)	ξ_1	0.2	3	0.286	0.286	0.0	2.1	0.0
(b)	ξ_1	1.0	3	0.291	0.291	0.0	0.4	0.0
(c)	ξ_1	4.0	3	0.292	0.292	0.0	0.1	0.0
(d)	ξ_1	6.0	3	0.292	0.292	0.0	0.1	0.0

The results are similar to the LPF case; only the first mode was entrained, and the errors are small, with e_ω decreasing with ω_o as predicted from Table I.

For BPF, we obtained the results below.

	ξ_ℓ	ω_o	η	ω_{MHB}	ω_{SIM}	e_{PRD}	e_ω	e_x
(a)	ξ_1	1.0	3	0.290	0.290	0.0	0.9	0.0
(b)	ξ_1	4.0	3	0.292	0.292	0.0	0.2	0.0
(c)	ξ_1	6.0	3	0.292	0.292	0.0	0.2	0.0
(d)	ξ_1	0.2	-3	0.303	0.303	0.0	3.5	0.0
(e)	ξ_2	1.0	-3	2.15	2.14	0.1	12	0.0
(f)	ξ_3	4.0	-3	7.49	7.48	0.1	45	0.0

As predicted from the mode-partition diagram in Fig. 1, only the first mode is entrained by a positive feedback, regardless of the value of ω_o. In contrast, every specified mode can be entrained by a negative feedback with an appropriately selected ω_o. For both positive and negative feedbacks, e_{PRD} and e_x are small, but e_ω is relatively large for (d)–(f). The deviation from ω_ℓ is predicted by the MHB method; the estimated frequency $\text{w}_\ell^{\pm}$ is close to ω_ℓ or ω_o when the damping ratio $\zeta_\ell := \rho\omega_\ell/2$ is small and the distance between ω_o and ω_ℓ is large.

V. Conclusion

We have investigated the possibility of achieving entrainment to a natural mode of oscillation for multi-DOF collocated mechanical systems using multi-agent controllers, each of which is placed in a feedback loop between a sensor/actuator pair. Necessary and sufficient conditions are obtained for the controller so that the harmonic balancing predicts a stable oscillation near a prescribed natural mode. The analytical results indicated, and numerical simulations confirmed, that multiple agents with simple dynamics are able to achieve natural entrainment, and that the band-pass filtering property of agent dynamics is important for entrainment to an arbitrarily specified mode. The mode-partition diagram for the BPF agent case suggests, however, that the BPF agent may not fully capture the dynamical properties of the CPG "agent" used in [8] that are important for *robust* entrainment against perturbations in mechanical parameters.

References

[1] F. Delcomyn. Neural basis of rhythmic behavior in animals. *Science*, 210:492–498, 1980.

[2] G. N. Orlovsky, T.G. Deliagina, and S. Grillner. *Neuronal Control of Locomotion: From Mollusc to Man*. Oxford University Press, 1999.

[3] T. Iwasaki and M. Zheng. Sensory feedback mechanism underlying entrainment of central pattern generator to mechanical resonance. *Biological Cybernetics*, 94(4):245–261, 2006.

[4] B. W. Verdaasdonk, H. F. Koopman, and F. C. van der Helm. Energy efficient and robust rhythmic limb movement by central pattern generators. *Neural Network*, 19(4):388–400, 2006.

[5] B. W. Verdaasdonk, H. F. Koopman, and F. C. van der Helm. Resonance tuning in a neuro-musculo-skeletal model of the forearm. *Biological Cybernetics*, 96(2):165–180, 2007.

[6] C. A. Williams and S. P. DeWeerth. A comparison of resonance tuning with positive versus negative sensory feedback. *Biological Cybernetics*, 96(6):603–614, 2007.

[7] Y. Futakata and T. Iwasaki. Formal analysis of resonance entrainment by central pattern generator. *J. Math. Biol.*, 57(2):183–207, 2008.

[8] Y. Futakata and T. Iwasaki. Entrainment of central pattern generators to natural oscillations of collocated mechanical systems. In *IEEE Conf. Decision and Contr.*, pages 5220–5225, Cancun, Mexico, December 2008.

[9] Y. Futakata. *Natural mode entrainment by CPG-based decentralized feedback controllers*. Ph.D. dissertation, Mechanical and Aerospace Engineering, University of Virginia, August 2009.

[10] T. Iwasaki. Multivariable harmonic balance for central pattern generators. *Automatica*, 44(12):4061–4069, 2008.

[11] Z. Chen, M. Zheng, W. O. Friesen, and T. Iwasaki. Multivariable harmonic balance analysis of neuronal oscillator for leech swimming. *J. Computational Neuroscience*, 25(3):583–606, 2008. (DOI: 10.1007/s10827-008-0105-7).

[12] V. A. Yakubovich. Frequency-domain criteria for oscillation in nonlinear systems with one stationary nonlinear component. *Siberian Mathematical Journal*, 14(5):768–788, 1973.

[13] A. Pogromsky, T. Glad, and H. Nijmeijer. On diffusion driven oscillations in coupled dynamical systems. *Int. J. Bifurcation and Chaos*, 9(4):629–644, 1999.